Extra Material for "Aggregate Savings When Individual Income Varies"
Martin Flodén, April 2, 2007

In Lemma 1 in Floden (2007) it is stated that $c_t \overset{a.s.}{\to} \frac{r}{1+r}\bar{W} - r\phi$. Chamberlain and Wilson (2000, theorem 3) and Ljungqvist and Sargent (2004, proposition 1 in chapter 16.3) state that $c_t \overset{a.s.}{\to} \frac{r}{1+r}\bar{W}$ in economies with no-borrowing constraints (i.e. $\phi = 0$). This document extends the proof to the case with a general ad hoc borrowing constraint $a_{t+1} \geq -\phi$.

Let $R = 1 + r$ denote the gross interest rate. In Floden (2007), the present value of future income is denoted by $W_t = \sum_{j=0}^{\infty} R^{-j} w z_{t+j}$. Define now $x_t = rR^{-1}W_t$, and $\bar{x} = \sup_t x_t$. The first part of Lemma 1 in Floden (2007) implies that $\bar{x} = rR^{-1}\bar{W}$, so the second part of that lemma can be reformulated as below.

Lemma 1

$$c_t \to \bar{x} - r\phi$$

Proof. Note first that the present-value budget constraint from date t and on is

$$\sum_{j=0}^{\infty} R^{-j} c_{t+j} \leq W_t + Ra_t \tag{1}$$

and that an optimal consumption plan implies that (1) holds with equality for all t.

We first show that $\lim c_t \leq \bar{x} - r\phi$. Suppose to the contrary that $\lim c_t > \bar{x} - r\phi$, and let τ be the smallest $\tau \geq 0$ such that $c_\tau > \bar{x} - r\phi$.[1] From the first order condition $u_c(c_{\tau-1}) \geq u_c(c_\tau)$, we then see that the borrowing constraint was binding at $\tau - 1$ so that $a_\tau = -\phi$ (or, if $\tau = 0$, we had $a_\tau = -\phi$ by assumption). Now, note that the first order condition implies that $c_{\tau+j} \geq c_\tau$ for all $j > 0$. Therefore,

$$\sum_{j=0}^{\infty} R^{-j} c_{\tau+j} \geq \sum_{j=0}^{\infty} R^{-j} c_\tau > \sum_{j=0}^{\infty} R^{-j} (\bar{x} - r\phi) \geq W_\tau + Ra_\tau$$

which violates the present-value budget constraint (1). It is therefore impossible that $c_t > \bar{x} - r\phi$. Since consumption is a non-decreasing sequence, we conclude that $\lim c_t \leq \bar{x} - r\phi$.

We next show that $\lim c_t \geq \bar{x} - r\phi$. Suppose to the contrary that $\lim c_t < \bar{x} - r\phi$. Then there is a τ such that $c_{\tau+j} < x_\tau - r\phi$ for all $j \geq 0$. This implies that

$$\sum_{j=0}^{\infty} R^{-j} c_{\tau+j} < \sum_{j=0}^{\infty} R^{-j} (x_\tau - r\phi) = W_\tau - R\phi \leq W_\tau + Ra_t$$

where the last inequality uses that the borrowing constraint implies $a_t \geq -\phi$. We thus see that the present-value budget constraint (1) does not hold with equality and thus that the consumption path is not optimal. This demonstrates that $\lim c_t \geq \bar{x} - r\phi$. ■

[1] Alternatively, if the limit does not exist the first order condition still implies that $c_{t+1} \geq c_t$ and we can again define τ as the smallest τ where $c_\tau > \bar{x} - r\phi$.

References

Chamberlain, Gary and Charles A. Wilson (2000), "Optimal Intertemporal Consumption under Uncertainty", *Review of Economic Dynamics*, 3, 365-395

Floden, Martin (2007), "Aggregate Savings When Individual Income Varies", manuscript, Stockholm School of Economics

Ljungqvist, Lars, and Thomas J. Sargent (2004), *Recursive Macroeconomic Theory*, Second Edition, MIT Press